EXERCISE – IV ADVANCED SUBJECTIVE QUESTIONS

1. Verify Rolle's theorem for the function

$f(x) = \log_e\left(\frac{x^2+ab}{x(a+b)}\right) + p$, for [a, b] where $0 < a < b$.

2. Using Rolle's theorem prove that the equation $3x^2 + px - 1 = 0$ has at least one real root in the interval (–1, 1).

3. If the equation $a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_{n-1} x = 0$ has a +ve root α, prove that the equation
$na_0 x^{n-1} + (n-1)a_1 x^{n-2} + \ldots\ldots\ldots + a_{n-1} = 0$ also has a positive root smaller than α.

4. Explain the failure of Lagrange's mean value theorem in the interval [–1, 1] for the function $f(x) = \frac{1}{x}$

5. If a, b are two real numbers with a < b show that a real number 'c' can be found between a and b such that $3c^2 = b^2 + ab + a^2$.

6. If a > b > 0, with the aid of Lagrange's formula, prove validity of the inequality
$nb^{n-1}(a-b) < a^n - b^n < na^{n-1}(a-b)$, if n > 1. Also prove that the inequalities are in opposite sense if $0 < n < 1$.

7. Using Rolle's theorem show that the derivative of the function $f(x) = \begin{cases} x\sin\frac{\pi}{x} & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$ vanishes at an infinite set of points of the interval (0, 1)

8. A function f is differentiable in the interval $0 \le x \le 5$ such that f(0) = 4 & f(5) = –1. If $g(x) = \frac{f(x)}{x+1}$, then prove that there exists some $c \in (0, 5)$ such that $g'(x) = -\frac{5}{6}$.

9. Let f(x) & g(x) be differentiable function so that $f(x)\, g'(x) \ne f'(x)\, g(x)$. Prove that between any two roots of f(x) there exist atleast one root of g(x).

10. f is continuous in [a, b] and differentiable in (a, b) (where a > 0) such that $\frac{f(a)}{a} = \frac{f(b)}{b}$. Prove that there exist $x_0 \in (a, b)$ such that $f'(x_0) = \frac{f(x_0)}{x_0}$.

11. Verify Rolles theorem for $f(x) = (x-a)^m (x-b)^n$ on [a, b]; m, n being positive integer.

12. Let f : [a, b] $\to$ R be continuous on [a, b] and differentiable on (a, b). If f (a) < f(b), then show that $f'(c) > 0$ for some $c \in (a, b)$.

13. Let $f(x) = 4x^3 - 3x^2 - 2x + 1$, use Rolle's theorem to prove that there exist c, 0 < c < 1 such that f(c) = 0.

14. Using LMVT prove that :

(a) tan x > x in $\left(0, \frac{\pi}{2}\right)$,

(b) sin x < x for x > 0

15. Prove that if f is differentiable on [a, b] and if f(a) = f(b) = 0 then for any real α there is an $x \in (a, b)$ such that $\alpha f(x) + f'(x) = 0$

16. For what value of a, m and b does the function

$f(x) = \begin{cases} 3 & x = 0 \\ -x^2 + 3x + a & 0 < x < 1 \\ mx + b & 1 \le x \le 2 \end{cases}$

satisfy the hypothesis of the mean value theorem for the interval [0, 2].

17. Suppose that on the interval [–2, 4] the function f is differentiable, f(–2) = 1 and $|f'(x)| \le 5$. Find the bounding functions of f on [–2, 4], using LMVT.

18. Let f, g be differentiable on R and suppose that f(0) = g(0) and $f'(x) \le g'(x)$ for all $x \ge 0$. Show that $f(x) \le g(x)$ for all $x \ge 0$.

19. Let f be continuous on [a, b] and differentiable on (a, b). If f(a) = a and f(b) = b then show that there exist distinct c_1, c_2 in (a, b) such that
$f'(c_1) + f'(c_2) = 2$.

20. Let f defined on [0, 1] be a twice differentiable function such that, $|f''(x)| \le 1$ for all $x \in [0, 1]$. If f(0)=f(1), then show that, $|f'(x)|<1$ for all $x \in [0, 1]$

21. f(x) and g(x) are differentiable functions for $0 \le x \le 2$ such that f(0) = 5, g(0) = 0, f(2) = 8, g(2) = 1. Show that there exists a number c satisfying 0 < c < 2 and $f'(c) = 3\, g'(c)$.

22. If f, ϕ, ψ are continuous in [a, b] and derivable in]a, b[then show that there is a value of c lying between a & b such that,

$$\begin{vmatrix} f(a) & f(b) & f'(c) \\ \phi(a) & \phi(b) & \phi'(c) \\ \psi(a) & \psi(b) & \psi'(c) \end{vmatrix} = 0$$

23. Show that exactly two real values of x satisfy the equation $x^2 = x \sin x + \cos x$.

24. Let a > 0 and f be continuous in [–a, a]. Suppose that f'(x) exists and $f'(x) \le 1$ for all $x \le (-a, a)$. If f(a) = a and f(–a) = –a, show that f(0) = 0.

25. Prove the inequality $e^x > (1 + x)$ using LMVT for all $x \in R_0$ and use it to determine which of the two numbers e^π and π^e is greater.